# Derivation of Kinematics used in DiffDrive class

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This document contains derivations for the kinematics of a differential drive robot with 2 conventional wheels. The body coordinate frame of the robot is located at the center between the 2 wheels. The distance between the two wheels (wheel base) is given by $2D$. The radius of the wheels is given by $r$. The configuration of the robot in the world frame is represented by $q = (\theta, x, y)$.

## 1 Relationship between wheel velocities and body twist

First we find the relationship between wheel velocities and body twist. For conventional wheels, the relationship between the control and the twist in the wheel frame is:

$$\begin{bmatrix} \dot{\theta} \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ r\dot{\phi} \\ 0 \end{bmatrix} \tag{1}$$

Where $\phi$ is the angular position of the wheel, and hence $\dot{\phi}$ is the rotational speed of the wheel in rad/s. Hence, the control to twist relationship for a 2-conventional wheel differential drive robot is:

$$\begin{bmatrix} \dot{\phi_1} \\ \dot{\phi_2} \end{bmatrix} = 1/r \begin{bmatrix} -D & 1 & 0 \\ D & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{v_x} \\ \dot{v_y} \end{bmatrix} \tag{2}$$

Where $\dot{\phi_1}$ is the rotational speed of the left wheel and $\dot{\phi_2}$ is the rotational speed of the left wheel, and $\dot{q} = (\dot{\theta}, \dot{x}, \dot{y})$ is the body twist of the robot.

The DiffDrive class implementation requires the following functionality:

- Track the configuration $(x, y, \theta)$ of a differential-drive robot
- Convert a desired twist to the equivalent wheel velocities required to achieve that twist
- Update the configuration of the robot, given updated wheel angles (assuming constant wheel velocity in-between updates)

### 1.1 Convert a desired twist to the equivalent wheel velocities required to achieve that twist

To calculate the required control input (wheel speeds) to achieve a desired body twist, we can write out the equations from (2):

$$\dot{\phi_1} = \frac{-D\dot{\theta} + v_x}{r} \tag{3}$$

$$\dot{\phi_2} = \frac{D\dot{\theta} + v_x}{r} \tag{4}$$

### 1.2 Calculate the body twist from a given control input

To find the resulting body twist from a given control input, we can solve equations (3) and (4) for the components of twist $\dot{\theta}$ and $\dot{v_x}$ in terms of the wheel speeds:

$$\dot{\theta} = \frac{r(\dot{\phi_2} - \dot{\phi_1})}{2D} \tag{5}$$

$$\dot{v_x} = \frac{r(\dot{\phi_1} + \dot{\phi_2})}{2} \tag{6}$$

$$\dot{v_y} = 0 \tag{7}$$

$v_y = 0$ since conventional wheels do not allow sliding in the $y$ direction.

## 2 Updating the robot's configuration from wheel angles

For a sufficiently short time interval, we can assume that wheel velocity has remained unchanged over the time interval. If we have a finite time interval $\Delta t$, we can rewrite equation 2 in terms of the change in wheel angle and change in configuration, mapping wheel velocities to twists.

$$\begin{bmatrix} \frac{\Delta\theta}{\Delta t} \\ \\ \frac{\Delta x_b}{\Delta t} \\ \\ \frac{\Delta y_b}{\Delta t} \end{bmatrix} = M \begin{bmatrix} \frac{\Delta\phi_1}{\Delta t} \\ \\ \frac{\Delta\phi_2}{\Delta t} \end{bmatrix} \tag{8}$$

Where $M$ is the pseudo-inverse of the $H$ matrix. We can see that $\Delta t$ does not factor into the equation as it cancels out on both sides.

Hence, from the current wheel angle we can find the change in wheel angles by comparing with the wheel angles recorded in the previous timestep. We find the body twist $V_b$, in the same vein as outlined in section 1.2 but using finite differences. We then integrate $V_b$ to get the transform between the previous body frame and the new body frame $T_{bb'}$, where the $b$ frame is the body frame of the current configuration stored in DiffDrive, and the $b'$ frame is the new (updated) body frame to be stored. Integration of a twist is implemented in the rigid2d library (in rigid2d.cpp) and makes use of the center of rotation concept.

Now, we need to update the configuration. The new configuration $q(k+1)$ can be computed as:

$$q(k+1) = q(k) + \Delta q \tag{9}$$

We see that this change in configuration expressed in the body frame $\Delta q_b$ is the change in the position and orientation of the body frame as expressed by the transform $T_{bb'}$, i.e.

$$T_{bb'} = T(\Delta\theta_b, \Delta x_b, \Delta y_b) = T(\Delta q_b) \tag{10}$$

We can then find the change in configuration required in equation 9 $\Delta q$ by multiplying the adjoint:

$$\Delta q = A(\theta, 0, 0)\Delta q_b \tag{11}$$

Where $\theta$ is the currently tracked theta configuration.
Hence, we can update the configuration according to equation 9 and save it into the member variables of the DiffDrive class.